CALCULATION OF A SHEET PILE WALL RELIABILITY INDEX IN ULTIMATE AND SERVICEABILITY LIMIT STATES

JERZY BAUER
Institute of Mining, Wrocław University of Technology,
Wybrzeże Wyspiańskiego 27, 50-370 Wrocław, Poland.

WOJIECH PULA, MAREK WYJADLOWSKI
Institute of Geotechnics and Hydrotechnics, Wrocław University of Technology,
Wybrzeże Wyspiańskiego 27, 50-370 Wrocław, Poland.

Abstract: The paper presents the procedure of calculating reliability index for a sheet pile wall loaded with a horizontal force. The procedure is carried out with the equations representing the ultimate and serviceability limit states. The calculation of the reliability index related to bearing capacity limit state was based on a response surface made of the largest bending moments in a sheet pile wall, while the reliability index of serviceability limit state was obtained by forming another response surface involving the maximum horizontal displacements of pile heads. The global reliability index was calculated with the use of the appropriate equations derived from system reliability analysis.

1. PRELIMINARY ASSUMPTIONS

Limit state design, which is in common use nowadays, involves the calculation of limit states of bearing capacity and serviceability. These calculations may be combined with reliability analysis resulting in probabilistic safety measures such as the probability of failure or the equivalent reliability index [1]. This kind of approach will be presented in this paper.

The calculations are made for a wall of reinforced concrete piles arranged in the form of a single row.

It has been decided that of all possible kinds of bearing capacity limit states, only the greatest bending moment, which may appear along the whole pile from its head to the base, will be analysed. The maximum bending moment cannot exceed the limit value $M_{ult}$ resulting from the adopted reinforcement degree. Exceeding the limit value $M_{ult}$ is considered a failure state. As regards the serviceability limit states, the horizontal displacement of pile head has been chosen from various kinds of these states. According to serviceability limit state principles, such a displacement cannot exceed a defined allowable value $u_{all}$. As in the previous case, exceeding the value $u_{all}$ is considered to be a failure state.

As a result of random variability of soil properties, these conditions assume the form of two random events:
\{M \leq M_{\text{ult}}\}, \ {u \leq u_{\text{all}}} \}
\tag{1}

where \(M\) stands for the bending moment at any cross-section of a pile, and \(u\) is a horizontal displacement of pile head. Failure-free state means that conditions (1) must be fulfilled. Taking account of the reliability analysis, it is then a two-component serial system and the problem consists in estimating the probability of failure:

\[P[(M \leq M_{\text{ult}}) \cap (u \leq u_{\text{all}})] . \]
\tag{2}

In a general case, the resulting failure probability

\[p_F = 1 - P[(M \leq M_{\text{ult}}) \cap (u \leq u_{\text{all}})] = P\{M > M_{\text{ult}} \cup (u > u_{\text{all}})\} \]
\tag{3}

is higher than the probabilities of failures obtained in the case where the conditions of bearing capacity and serviceability limit states are not met separately.

2. NUMERICAL ANALYSIS OF THE WORK OF A HORIZONTALLY-LOADED SHEET PILE WALL

The aim of a numerical analysis based on the computational model shown in figure 1 was to obtain a set of maximum bending moments in the sheet pile wall and of pile head horizontal displacements for the given values \(E_1\) and \(E_2\) of Young’s modulus in the upper and middle layers of soil, respectively. These values will make it possible to find the response surfaces for the bending moment and for displacement. The computational FEM model presented in figure 2 incorporates three layers of soil.

In the first two layers, 4 m deep each, the elastic moduli \(E_1\) and \(E_2\) are treated as random variables. The third layer of a good quality ground is assumed to be homogeneous with non-random elastic modulus \(E_3\) (see figure 1). The pile sheet wall with the mean thickness of 0.44 m reaches the depth of 9 m and is transversely loaded with the force \(P = 44\) kN/m, applied to the pile head. In figure 2, two vertical lines near the wall limit the area with a higher number of triangular six-node finite elements.

Two assumptions have been imposed on the numerical model. The first is that the first layer is completely saturated, while the second assumption is that the loading is repeatedly applied and acts for a short period of time. The assumption that the loading acts in an almost pulsing way in saturated soil makes it possible to assume full contact between the pile and the surrounding soil on the left-hand side of the wall where the tensile stresses appear. The calculations confirm that the value of these tensile stresses do not exceed 0.025 MPa, i.e. 1/4 atmospheric pressure. If the loading in saturated soil changes quickly, the pile movement causes suction to act on the left-hand side of the wall. This justifies the assumption of a complete connection between the soil and the sheet pile wall.
Fig. 1. Sheet pile wall embedded in a layered soil and subjected to horizontal force

Fig. 2. A computational finite element model
Fig. 3. Compressive stresses in extreme pile fibres

Fig. 4. Horizontal displacement stresses along sheet pile wall
The curve in figure 3 presents the compressive stress $\sigma_y$ in the extreme fibers of the piles. Relation

$$M = W \sigma_y$$  \hspace{1cm} (4)

enables the calculation of bending moments, where $W$ is the bending index corresponding to one linear metre of the wall: $W = 1 \cdot (0.44)^2 / 6$. Figure 4 presents the horizontal displacements of the wall. The maximum displacement corresponds to the displacement at the pile head.

The characteristics of random variables corresponding to material properties used in the numerical computations are gathered in table 1.

### Table 1

<table>
<thead>
<tr>
<th>Layer types</th>
<th>Mean values of Young’s modulus $E$ (MPa)</th>
<th>Values of Poisson’s ratio $\nu$</th>
<th>Standard deviations of Young’s modulus $\sigma_E$ (MPa)</th>
<th>Probability distributions of Young’s modulus $E$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Upper layer</td>
<td>4</td>
<td>0.30</td>
<td>0.45</td>
<td>lognormal</td>
</tr>
<tr>
<td>Middle layer</td>
<td>20</td>
<td>0.30</td>
<td>3.00</td>
<td>lognormal</td>
</tr>
<tr>
<td>Bottom layer</td>
<td>80</td>
<td>0.35</td>
<td>0</td>
<td>deterministic</td>
</tr>
<tr>
<td>Concrete sheet pile wall</td>
<td>28 000</td>
<td>0.16</td>
<td>0</td>
<td>deterministic</td>
</tr>
</tbody>
</table>

The computations were carried out with the use of the finite element software PHSESES2, specially designed for analysing the stresses and displacements of underground structures in plain strain problems.

### 3. ULTIMATE AND SERVICEABILITY LIMIT STATES OF A PILE SHEET WALL

According to Polish Standard [2] the correctness of the structure designed should be checked through analysing the limit states of bearing capacity and serviceability. As the pile sheet wall will be subjected to bending, the requirement of not exceeding the bearing capacity in the most loaded cross-section implies the first inequality in (1). The values of the ultimate bending moment $M_{\text{ult}}$ depend on the percentage of reinforcement in a particular cross-section (table 2). These values are determined with a simplified method that is based on normative correlations for the cross-section height $h = 0.44$ m, the concrete cover thickness $a = 0.05$ m and the design plasticity limit of steel plasticity $f_p = 210$ MPa.
Table 2

Values of boundary moment depending on reinforcement percentage

<table>
<thead>
<tr>
<th>Reinforcement percentage (%)</th>
<th>$A_s$ (cm²/m)</th>
<th>$A_{s1}$ (cm²/m)</th>
<th>$M_{ult}$ (kNm/m)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.50</td>
<td>22.00</td>
<td>11.00</td>
<td>78.54</td>
</tr>
<tr>
<td>0.55</td>
<td>24.20</td>
<td>12.10</td>
<td>86.40</td>
</tr>
<tr>
<td>0.60</td>
<td>26.40</td>
<td>13.20</td>
<td>94.25</td>
</tr>
<tr>
<td>0.65</td>
<td>28.60</td>
<td>14.30</td>
<td>102.10</td>
</tr>
</tbody>
</table>

$A_s$ – the reinforcement surface, $A_{s1}$ – the surface of tensioned reinforcement, $M_{ult}$ – the maximum value of bending moment.

The condition of not exceeding the boundary value of the horizontal displacement $u_{all} = 1.1$ cm by pile heads produces the second inequality in formula (1).

In the next section of this paper, it will be assumed that the bending moment $M$ and the horizontal displacement $u$ are the functions of two random values of the elastic modulus $E$ and the random horizontal force $P$.

## 4. RESPONSE SURFACES FOR BENDING MOMENT AND DISPLACEMENT

In order to obtain the reliability measures, the functional relationships between ‘input’ random variables (the values of elastic modulus) and those obtained ‘at the output’ of the problem (bending moment, head displacement) should be known. Unfortunately, no explicit functional relationships between these quantities are known and it is only possible to define, by means of finite element method, the set of relevant physical quantities for the adopted values of material constants. In this situation, the response surface method was used to calculate the reliability indices. The method generally consists in approximating an unknown function, for which only a certain amount of values is known, with an adequately adopted function. The choice of approximating function may be based on the results of experimental research as well as on the results of numerical calculations, e.g. the results obtained by finite element method. A detailed information can be found in numerous monographs, e.g. that by MYERS and MONTGOMERY [3].

This paper uses an iterative algorithm based on nonlinear regression and the Marquardt method, adapted by BAUER and PULA [4]. In the case discussed, the response surface has the form of a second degree polynomial in two variables, i.e.:

$$f(E_1, E_2) = \frac{P}{44} (b_1 + b_2 E_1 + b_3 E_2 + b_4 E_1^2 + b_5 E_2^2 + b_6 E_1 E_2) + \varepsilon,$$

where $\varepsilon$ is the random variable defining a fitting error (cf. BAUER and PULA [4]), $E_1$, $E_2$ are random values of Young’s modulus and $b_1$–$b_6$ – the coefficients of the desired
Calculation of a sheet pile wall reliability index

The function $f$ was determined fivefold – once in the case of approximating the displacement $u$ and four times – for approximating the maximum bending moment $M$, depending on the boundary value (c.f. table 2). The values $B_1$–$B_6$ of coefficients $b_1$–$b_6$ obtained in particular cases are presented in table 3.

### Table 3

<table>
<thead>
<tr>
<th>Coefficients of response surfaces</th>
<th>$u = 1.1$ (cm)</th>
<th>$M_{\text{bound}} = 78.54$ (kNm/m)</th>
<th>$M_{\text{bound}} = 86.40$ (kNm/m)</th>
<th>$M_{\text{bound}} = 94.25$ (kNm/m)</th>
<th>$M_{\text{bound}} = 102.1$ (kNm/m)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$B_1$</td>
<td>2.556</td>
<td>115.77</td>
<td>132.851</td>
<td>136.391</td>
<td>148.8</td>
</tr>
<tr>
<td>$B_2$</td>
<td>-0.405</td>
<td>-23.7828</td>
<td>-32.5212</td>
<td>-33.3744</td>
<td>-37.27</td>
</tr>
<tr>
<td>$B_3$</td>
<td>-0.0472</td>
<td>1.61518</td>
<td>1.33108</td>
<td>1.13407</td>
<td>0.6371</td>
</tr>
<tr>
<td>$B_4$</td>
<td>0.0273</td>
<td>1.75618</td>
<td>3.03030</td>
<td>3.11103</td>
<td>3.722</td>
</tr>
<tr>
<td>$B_5$</td>
<td>0.616E-3</td>
<td>-0.02001</td>
<td>-0.0148148</td>
<td>-0.0116667</td>
<td>-0.4866E-2</td>
</tr>
<tr>
<td>$B_6$</td>
<td>0.324E-2</td>
<td>-0.022728</td>
<td>-0.674E-9</td>
<td>0.016661</td>
<td>0.04807</td>
</tr>
</tbody>
</table>

### 5. Computing reliability measures

Reliability is calculated in two steps. The first step is to calculate the probabilities of failure

$$p_{F_1} = P\{M > M_{\text{ult}}\}$$

connected successively with four values $M_{\text{ult}}$ of boundary bending moment and the reliability indices $\beta_1$ related to these probabilities by the following equation

$$p_{F_1} = \Phi_0(-\beta),$$

where $\Phi_0$ is the cumulative normal distribution function.

The second step is to calculate the probability

$$p_{F_2} = P\{u > u_{\text{all}}\}$$

connected with exceeding allowable horizontal displacements and the reliability index $\beta_2$ related to this probability (equation (7)). These measures were calculated by means of the SORM method, widely used in the reliability theory (cf. HOHENBICHLER et al. [5] or PULA [6]). The reliability indices $\beta_1$ corresponding to the probability (6) of not meeting the condition of ultimate limit state and the reliability indices $\beta_2$ corresponding to probability (8) of not meeting the condition of serviceability limit state can be found in table 4.
Table 4

Values of reliability indices depending on reinforcement percentage

<table>
<thead>
<tr>
<th>Reinforcement percentage (%)</th>
<th>$\beta_1$</th>
<th>$\beta_2$</th>
<th>$\beta$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.50</td>
<td>1.41</td>
<td>2.31</td>
<td>1.38</td>
</tr>
<tr>
<td>0.55</td>
<td>2.90</td>
<td>2.31</td>
<td>2.27</td>
</tr>
<tr>
<td>0.60</td>
<td>4.37</td>
<td>2.31</td>
<td>2.31</td>
</tr>
<tr>
<td>0.65</td>
<td>5.83</td>
<td>2.31</td>
<td>2.31</td>
</tr>
</tbody>
</table>

However, as mentioned in section 1, our aim was to determine the probability of failure:

$$p_F = P\{(M > M_{ult}) \cup (u > u_{ult})\},$$

and the global reliability index $\beta$ related to this probability through equation (7). The method for determining the global reliability index, one of the problems included in system reliability analysis, is very briefly described below.

In practice, in order to fulfil the safety conditions, a structure must often meet several various, often interrelated, criteria. Therefore one should adopt a model whose structure is the system of components, each of them satisfying different failure criteria characterised by different limit state functions. These elements can form a series system (like in the example discussed in this paper) – then the failure-free operation of the whole system requires failure-free operation of all the elements. The opposite of the system of components is a parallel system, where a lack of failure in one element is sufficient for the failure-free operation of the whole system. A mixed system is a combination of the parallel and serial systems. In calculating the probability of such a system it is usually necessary to estimate the probability of the following event:

$$F_{sys} = \bigcup_i \bigcap_j \{g_{ij}(X) \leq 0\},$$

where $g_{ij}$ stands for the limit state function of the element $ij$. Therefore it is necessary to estimate the probability of the union and intersection of events. If the events analysed are not mutually exclusive, the so-called Ditlevsen’s bounds are often applied (DITLEVSEN [7]):

$$P\left(\bigcup_{i=1}^{q} F_i\right) = \left\{ \begin{array}{l} 
\leq \sum_{i=1}^{q} P(F_i) - \sum_{i=2}^{q} \max\{P(F_i \cap F_j)\} \leq \sum_{i=1}^{q} P(F_i), \\
\geq P(F_1) + \sum_{i=2}^{q} \max\left\{0, P(F_i) - \sum_{j=1}^{i-1} \{P(F_i \cap F_j)\}\right\} \geq \max_{i=1}^{q} P(F_i) \geq 0. 
\end{array} \right.$$
If the sum on the right-hand side of inequality (11) is higher than unity, the upper limit is, of course, unity. As we can see, the use of inequality (11) requires calculating the probabilities of intersections in the form of \( P(F_i \cap F_j) \). These probabilities can be estimated after employing transformation for standard normal space. It can be proved that for two-dimensional cumulative normal distribution function \( \Phi_2 \) with mean \((0, 0)\) and variance \((1, 1)\) and correlation coefficient between the components \( \rho \), the following equality is valid:

\[
\Phi_2(x, y, \rho) = \Phi_0(x)\Phi_0(y) + \int_0^\rho \phi_2(x, y, t)dt,
\]

where \( \phi_2 \) stands for the density corresponding to \( \Phi_2 \) (cf. [1]). This equality leads to the following approximate formula:

\[
P(F_i \cap F_j) \approx \Phi_0(-\beta_i)\Phi_0(-\beta_j) + \int_0^{\rho_{ij}} \phi_2(-\beta_i, -\beta_j, t)dt,
\]

where \( \rho_{ij} \) stands for the coefficient of correlation between the random variables \( g_i(X) \) and \( g_j(X) \), while \( g_i, g_j \) are limit state functions for the elements \( i \) and \( j \), respectively, and \( \beta_i \) and \( \beta_j \) – the corresponding reliability indices according to FORM method.

When applying the above-mentioned Ditlevsen’s bounds, the probability (9) corresponding to four cases of reinforcement surfaces was calculated as well as the global reliability indices \( \beta \) were listed in table 4.

### 6. FINAL REMARKS

When analysing the computation results of the reliability given found in table 4, one can describe three different situations. The reliability indices in the first row of table 4 refer to the case of a relatively low value of boundary moment resulting from a small percentage of reinforcement. The possibility of the system failure is connected mostly with the probability of exceeding the condition of ultimate limit state with the reliability index \( \beta_1 = 1.41 \). This probability of failure is slightly increased towards the global probability of failure due to additional possibility of exceeding the condition of serviceability limit state with the reliability index \( \beta_2 = 2.31 \), which produces the value \( \beta = 1.38 \) of the global reliability index. In the case presented in the second row of table 4, the situation is opposite. The system failure chiefly does not meet the condition of serviceability limit state. The increased value of boundary moment reduces the probability of failure resulting from not meeting the condition of bearing capacity limit state, which is reflected in the increase of reliability index to the value \( \beta_1 = 2.90 \), and only slightly reduces the value of global reliability index – from the value \( \beta_2 = 2.31 \) related to exceeding the condition of serviceability limit state to the global value \( \beta = 2.27 \).
The third case, shown in rows 3 and 4 of table 4, is different. System failure is chiefly caused by exceeding the condition of serviceability limit state, and the global reliability index $\beta$ has the same value as the coefficient $\beta_2 = 2.31$ related to serviceability limit state. This means that there is practically no stochastic interaction between the conditions of ultimate limit states and serviceability limit states. This situation results from the fact that the failure area related to exceeding boundary moment lies far from the origin of the coordinate system of standardized random variables of the material constants $E_1$ and $E_2$. One should note that the third case is the most advantageous if a structural safety is taken into account.

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